Necessary and Sufficient Detection Efficiency for the Mermin Inequalities

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We prove that the threshold detection efficiency for a loophole-free Bell experiment using an n-qubit Greenberger-Horne-Zeilinger state and the correlations appearing in the n-partite Mermin inequality is $n/(2n-2)$. If the detection efficiency is equal to or lower than this value, there are local hidden variable models that can simulate all the quantum predictions. If the detection efficiency is above this value, there is no local hidden variable model that can simulate all the quantum predictions.

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Quantum nonlocality is the impossibility of reproducing the quantum correlations between the results of distant measurements using local hidden variable (LHV) theories. This impossibility is shown either by the violation of a Bell inequality [1] or the impossibility of ascribing predefined results which simultaneously satisfy several predictions of quantum mechanics [2]. Although quantum nonlocality is intimately related to entanglement [3], security of quantum cryptography [4], and communication complexity [5], there is as of yet no loophole-free quantum nonlocality experiment. A particulary important problem is the detection loophole. It occurs when the imperfect efficiency of the detectors leaves room for LHV theories in which undetected events can occur due to local hidden instructions rather than to imperfections [6]. An appropriate measure of the quantum nonlocality of a given quantum state and Bell inequality is, therefore, the minimum detection efficiency required for a loophole-free Bell experiment, $\eta_{\rm crit}$. It is defined as the value of the ratio between detected and emitted particles such that, if $\eta \leq \eta_{\rm crit}$, there is a LHV theory reproducing the predictions of quantum mechanics, but no such LHV theories exist if $\eta > \eta_{\rm crit}$. The value of $\eta_{\rm crit}$ is known for some scenarios [7, 8, 9, 10, 11, 12], and some general bounds have been obtained [13]. Curiously, $\eta_{\rm crit}$ was still unknown for a very important scenario.

Eighteen years ago, Mermin and others discovered the first example of a Bell inequality with a violation that grows exponentially with the number n of particles [14, 15]. Specifically, they show that the n-qubit Greenberger-Horne-Zeilinger (GHZ) state $|{\rm GHZ}_n\rangle$ [2] violates a n-partite Bell inequality by an amount that grows as $2^{(n-1)/2}$ [14, 15]. If instead of a pure $|{\rm GHZ}_n\rangle$ we have a noisy one, $V|{\rm GHZ}_n\rangle\langle{\rm GHZ}_n| + (1-V)\mathbb{1}/2^n$, then the minimum value of V required to observe a violation is $V_{\rm crit} = 2^{(1-n)/2}$. Later, Werner and Wolf proved that the Mermin inequality is the two-setting correlation Bell inequality "which can be violated by the widest margin in quantum theory . . . [and] is the only one for which the maximal violation $2^{(n-1)/2}$ is attained" [16].

There have been several attempts to obtain $\eta_{\rm crit}$ for the Mermin inequality: Braunstein and Mann showed that $\eta_{\rm crit} \geq 2^{(1-n)/2n}$ for n odd, and $\eta_{\rm crit} \geq 2^{(2-n)/2n}$ for n even [17]; Brassard, Broadbent, and Tapp showed that $\eta_{\rm crit} < 2^{(2-n)/n}$ [18]; and Larsson proved that $\eta_{\rm crit} = 3/4$ for $n = 3$ [9]. However, no formula was known for arbitrary n [19]. In this Letter we prove that $\eta_{\rm crit} = n/(2n-2)$. In addition, we obtain numerically the relation between $\eta_{\rm crit}$ and $V_{\rm crit}$ for several values of n.

The Mermin inequality is based on the GHZ proof of Bell's theorem [2]. It shows the impossibility of assigning predefined values -1 or 1 to local observables, simultaneously satisfying several perfect correlations predicted by quantum mechanics. The scenario for the GHZ proof is the following. A system composed of $n \geq 3$ particles is initially prepared in the state $|{\rm GHZ}_n\rangle$. Each particle moves away to a distant space-time region where an observer measures randomly either X_i or Z_i, where X and Z denote the Pauli matrices σ_x and σ_z, and i denotes particle i. Local measurements on particle i are assumed to be spacelike separated from the choices of measurements made on all other particles.

The n-qubit GHZ state is the unique simultaneous eigenstate that satisfies

$$g_i|{\rm GHZ}_n\rangle = |{\rm GHZ}_n\rangle, \quad \text{for } i = 1, \ldots, n, \tag{1}$$

where

$$g_i = X_i \bigotimes_{j\neq i}^{n} Z_j \tag{2}$$

are the generators of the stabilizer group of the GHZ state, defined as the set $\{s_j\}_{j=1}^{2^n}$ of all products of the generators. The perfect correlations of the GHZ state are

$$\langle{\rm GHZ}_n|s_j|{\rm GHZ}_n\rangle = 1, \quad \text{for } j = 1, \ldots, 2^n. \tag{3}$$

For n odd, the GHZ proof is as follows. Any LHV theory assigning predefined values -1 or 1 to X_i and Z_i

in agreement with the quantum predictions given by (3) must satisfy

$$s_j = 1, \quad \text{for } j = 1, \ldots, 2^n. \tag{4}$$

However, if we take into consideration only the 2^{n-1} predictions (4) involving stabilizing operators s_j that are products of an odd number of generators, and assume predefined values either -1 or 1, then it so happens that, at most, only $2^{n-2} + 2^{(n-3)/2}$ out of these 2^{n-1} predictions are satisfied. For the remaining predictions of quantum mechanics, the corresponding prediction of the LHV model is the opposite (i.e., $s_j = -1$); the reason for this behavior will be explained below [see (ii)]. Therefore, this discrepancy between quantum mechanics and LHV theories can be reformulated as a violation of a Bell inequality. Any LHV theory must satisfy the following inequality:

$$|\beta_n| \leq 2^{(n-1)/2}, \tag{5}$$

where the Bell operator

$$\beta_n = \frac{1}{2}\left[\prod_{i=1}^{n}(\mathbb{1} + g_i) - \prod_{i=1}^{n}(\mathbb{1} - g_i)\right] \tag{6}$$

is the sum of all stabilizing operators which are products of an odd number of generators. Inequality (5) is the Mermin inequality for n odd [14]. On the other hand, the n-qubit GHZ state satisfies

$$\langle \mathrm{GHZ}_n|\beta_n|\mathrm{GHZ}_n\rangle = 2^{n-1}, \tag{7}$$

and therefore violates the Mermin inequality (5) by an amount that grows as $2^{(n-1)/2}$ [14]. For n even, the Mermin inequality is not violated by $2^{(n-1)/2}$. The equivalent inequality was found by Ardehali [15]. For an explanation of the Ardehali inequality in terms of stabilizers of the GHZ state, see [20]. For simplicity's sake, we will focus on the Mermin inequality for n odd. Our proof works similarly for n even when we consider the Ardehali inequality.

Following [10], we include the detector inefficiency in the LHV model, so that the model consists of a set of instructions telling the n particles what to do if X or Z are measured. For a given particle, the only possible instructions are "give a detection (-1 or 1)" or "do not give a detection."

$P(X_i)$ is the probability that particle i is detected (giving either -1 or 1) when X_i is measured. $P(X_i|X_j)$ is the probability that particle i is detected when X_i is measured if particle $j \neq i$ is detected when X_j is measured. $P(X_i|X_jZ_k)$ is the probability that particle i is detected when X_i is measured if particle j ($j \neq i$) is detected when X_j is measured and particle k ($i \neq k \neq j \neq i$) is detected when Z_k is measured. Analogously, $P(X_iZ_j|Z_k)$ is the probability that particle i is detected when X_i is measured and particle j ($j \neq i$) is detected when Z_j is measured if particle k ($i \neq k \neq j \neq i$) is detected when Z_k is measured.

In our LHV models, measurement results are predefined and are independent of the measurements on other particles. In addition, they must satisfy some restrictions dictated by the expected (and testable) behavior of the detectors and the properties of the n-qubit GHZ state for the measurements involved in a test of the Mermin inequality. Specifically, the following assumptions lead to the following restrictions.

(i) *All detectors have equal, constant detection efficiency. The efficiency is the same when X or Z are measured. The detection errors are independent. The detectors have no dark counts.*—From these assumptions it follows that $P(A_i) = p$, $\forall A \in \{X, Z\}$ and $\forall i \in \{1, 2, \ldots, n\}$, and $P(A_i, \ldots, B_j|C_k, \ldots, D_l) = p^r$, $\forall A, \ldots, B, C, \ldots, D \in \{X, Z\}$ and $\forall$ (different) $i, \ldots, j, k, \ldots, l \in \{1, 2, \ldots n\}$; r is the number of elements in $A_i, \ldots, B_j$. That is, the different probabilities must be symmetric under particle permutation and under the permutation of X_i and Z_i. If p is the minimum over all possible LHV models, then $\eta_{\text{crit}} = p$.

(ii) *Compatibility with the statistical predictions of quantum mechanics for the Mermin inequality using the n-qubit GHZ state.*—Each of the terms obtained by expanding (6), e.g., $X_1Z_2Z_3\ldots Z_n$, represents an experimental configuration required to test inequality (5). We will also consider configurations obtained for the previous ones by selecting measurements on subsets containing an odd number of particles, e.g., $X_1Z_2Z_3$. According to the predictions of quantum mechanics for the n-qubit GHZ state, in each of these experimental configurations, when an odd number $3 \leq q \leq n$ of particles are detected, the corresponding results must satisfy

$$\begin{aligned} X_iZ_jZ_k\cdots Z_q &= Z_iX_jZ_k\cdots Z_q = Z_iZ_jX_k\cdots Z_q = \cdots \\ &= Z_iZ_jZ_k\cdots X_q = -X_iX_jX_k\cdots X_q. \end{aligned} \tag{8}$$

In addition, if $q \neq n$, then (8) must equal $Z_{q+1}\ldots Z_n$. Therefore, depending on the result, -1 or 1, of the product $Z_{q+1}\ldots Z_n$, we can divide the reduced state of the q particles in two ensembles. For each of these ensembles a different GHZ proof applies. If $q = n$, then (8) must equal 1. Since these conditions cannot be fulfilled if X and Z of three or more particles have predefined values either -1 or 1, then we will conclude that the only hidden instructions allowed in the LHV model are those in which X and Z of three different particles have not all of them predefined values.

The challenge is to obtain the maximum possible detection efficiency that can be reproduced with LHV models which satisfy (i) and (ii). Each of these LHV models is defined on a probability space (Λ, ρ), and is made up of subsets of instructions $I_{k,l,m} \subset \Lambda$, each of them characterized by three numbers: k is the number of particles for

which both observables (X and Z) are predefined (i.e., would give a detection when the observable is measured), l is the number of particles for which only one of the observables (X or Z) is predefined, and $m = n - l - k$ is the number of particles for which none of the observables are predefined.

Five lemmas are needed to prove our main result.

Lemma 1.—In order to find the maximum detection efficiency that can be reproduced with LHV models which satisfy (i), it suffices to consider LHV models where each of the subsets $I_{k,l,m}$ satisfies (i).

Proof.—Suppose we find a LHV model compatible with a detection efficiency η, such that some of the subsets $I_{k,l,m}$ do not satisfy (i). Since the model must satisfy (i), we can always symmetrize it in all possible ways (by changing Z's to X's, and interchanging the different particles) and consider an average of all these rearrangements. Clearly, this new model will have the same η and each of the $I_{k,l,m}$ will satisfy (i). The new model will satisfy (ii) if and only if the original model satisfied (ii). ■

Therefore, from now on we will only consider models such that each of the $I_{k,l,m}$ satisfies (i). Each subset $I_{k,l,m} \subset \Lambda$ occurs with probability $0 \leq \rho_{k,l,m} \leq 1$.

In order to satisfy (ii), the only subsets of instructions $I_{k,l,m}$ allowed in our LHV models are those with $k = 0, 1, 2$. In addition, the predefined values must satisfy (8), and the -1 and 1 values must be suitably distributed in order to reflect the fact that for the GHZ state all of the one qubit reduced density matrices are maximally mixed. Notice that these last two conditions are not particularly restrictive and can be easily satisfied. Therefore, in order to improve the clarity of the presentation, we will not insist on them hereafter.

An upper bound on η will follow from probabilistic considerations on each of the $I_{k,l,m}$. We will use the notation $P_{I_{k,l,m}}$ to refer to the probabilities of detection of the different variables within the sets $I_{k,l,m}$.

Lemma 2.—The value of $P_{I_{2,n-2,0}}(X_1|X_2,\ldots,X_n)$ (and all the possible substitutions of X_i by Z_i and permutations of the indexes) is $n/(2n-2)$.

Proof.—By definition,

$$P_{I_{2,n-2,0}}(X_1|X_2,\ldots,X_n) = \frac{P_{I_{2,n-2,0}}(X_1,\ldots,X_n)}{P_{I_{2,n-2,0}}(X_2,\ldots,X_n)}. \quad (9)$$

In the subset $I_{2,n-2,0}$, only $\binom{n}{2}$ instructions have predefined values for all the X_i's. Since the total number of different instructions in $I_{2,n-2,0}$ is $\binom{n}{2}2^{n-2}$, then

$$P_{I_{2,n-2,0}}(X_1,\ldots,X_n) = \frac{1}{2^{n-2}}. \quad (10)$$

In order to calculate $P_{I_{2,n-2,0}}(X_2,\ldots,X_n)$, we consider the subset $S \subset I_{2,n-2,0}$ where both X_1 and Z_1 have predefined values. We also consider the complementary subset $S^c = I_{2,n-2,0} \setminus S$. Clearly, $P_{I_{2,n-2,0}}(S) = \frac{\binom{n-1}{1}}{\binom{n}{2}}$ and $P_{I_{2,n-2,0}}(S^c) = \frac{\binom{n-1}{2}}{\binom{n}{2}}$. Reasoning in S and S^c as above, we see that

$$\begin{aligned} &P_{I_{2,n-2,0}}(X_2,\ldots,X_n) \\ &= P_{I_{2,n-2,0}}(X_2,\ldots,X_n|S)P_{I_{2,n-2,0}}(S) \\ &+ P_{I_{2,n-2,0}}(X_2,\ldots,X_n|S^c)P_{I_{2,n-2,0}}(S^c) \\ &= \frac{1}{2^{n-2}}\frac{\binom{n-1}{1}}{\binom{n}{2}} + \frac{1}{2^{n-3}}\frac{\binom{n-1}{2}}{\binom{n}{2}} = \frac{2n-2}{n2^{n-2}}. \end{aligned} \quad (11)$$

■

Lemma 3.—For every $I_{k,l,m}$ different than $I_{2,n-2,0}$, $P_{I_{k,l,m}}(X_1|X_2,\ldots,X_n)$ is either undefined or less than $n/(2n-2)$.

Proof.—If $m > 1$, $P_{I_{k,l,m}}(X_1|X_2,\ldots,X_n)$ is not defined. If $m = 1$, $P_{I_{k,l,m}}(X_1|X_2,\ldots,X_n) = 0$; hence, we consider only the case $m = 0$. In this case, there are only two subsets to consider, $I_{1,n-1,0}$ and $I_{0,n,0}$.

Reasoning as in the proof of Lemma 2,

$$P_{I_{0,n,0}}(X_1|X_2,\ldots,X_n) = \frac{1}{2} \quad (12)$$

and

$$P_{I_{1,n-1,0}}(X_1|X_2,\ldots,X_n) = \frac{n}{2n-1}, \quad (13)$$

which is always less than $\frac{n}{2n-2}$. ■

Lemma 4.—For efficiencies higher than $\eta = n/(2n-2)$, there are no LHV models which simultaneously reproduce all the quantum predictions (8) for q odd and $3 \leq q \leq n$.

Proof.—The LHV model must satisfy $\eta = P(X_1) = P(X_1|X_2,\ldots,X_n)$. The value of $P(X_1|X_2,\ldots,X_n)$ must be less than or equal to the maximum of the values, where defined, of $P_{I_{k,l,m}}(X_1|X_2,\ldots,X_n)$ for the different subsets $I_{k,l,m}$ of the LHV model. According to Lemmas 2 and 3, all of these values are less than $n/(2n-2)$. ■

Lemma 5.—For $\eta = n/(2n-2)$, there are LHV models which simultaneously reproduce all the quantum predictions (8) for q odd and $3 \leq q \leq n$.

Proof.—We prove it by constructing explicit LHV models $M_n(\eta) = \{(\rho_{k,l,m}, I_{k,l,m})\}$ reproducing the quantum predictions for a given n and η. Exact LHV models for $n = 3, 4, 5$ for $\eta = n/(2n-2)$ are the following:

$$M_3\left(\tfrac{3}{4}\right) = \left\{\left(\tfrac{54}{64}, I_{2,1,0}\right), \left(\tfrac{9}{64}, I_{1,0,2}\right), \left(\tfrac{1}{64}, I_{0,0,3}\right)\right\}, \tag{14}$$

$$M_4\left(\tfrac{2}{3}\right) = \left\{\left(\tfrac{64}{81}, I_{2,2,0}\right), \left(\tfrac{8}{81}, I_{2,0,2}\right), \left(\tfrac{8}{81}, I_{1,0,3}\right), \left(\tfrac{1}{81}, I_{0,0,4}\right)\right\}, \tag{15}$$

$$M_5\left(\tfrac{5}{8}\right) = \left\{\left(\tfrac{25000}{2^{15}}, I_{2,3,0}\right), \left(\tfrac{3750}{2^{15}}, I_{2,1,2}\right), \left(\tfrac{1750}{2^{15}}, I_{2,0,3}\right), \left(\tfrac{2025}{2^{15}}, I_{1,0,4}\right), \left(\tfrac{243}{2^{15}}, I_{0,0,5}\right)\right\}, \tag{16}$$

where, e.g., $\left(\frac{54}{64}, I_{2,1,0}\right)$ means that the model has instructions $I_{2,1,0}$ with probability $\frac{54}{64}$, etc. For higher n, we have obtained LHV models for $\eta = n/(2n-2)$ numerically for up to $n = 15$ qubits. For a given n and $\eta = n/(2n-2)$, the LHV models are not unique.

In addition, we have calculated numerically the maximum background noise [8, 12] as a function of the minimum detection efficiency required to violate the Mermin inequality. The results, for up to $n = 8$ qubits, are summarized in Fig. 1.

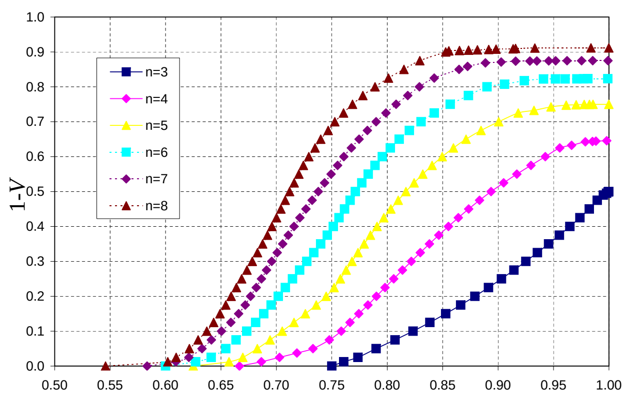

FIG. 1: Maximum (background) noise $1-V$ as a function of the minimum detection efficiency η required to violate the Mermin inequality, when the state is $V|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n| + (1-V)\mathbb{1}/2^n$, for $n = 3, 4, \ldots, 8$ qubits. Note that if $V = 1$, then $\eta = n/(2n-2)$, and if $\eta = 1$, then $V = 2^{(1-n)/2}$.

We have proven that a loophole-free Bell experiment using an n-qubit GHZ state and the correlations appearing in the n-partite Mermin inequality requires a detection efficiency higher than $n/(2n-2)$. This result solves a long-standing open problem and is specially relevant for the 4 [21], 5 [22], and 6-qubit GHZ states [23] prepared in recent experiments. $n/(2n-2)$ is the threshold efficiency beyond which there is no LHV model which simultaneously satisfies all the quantum predictions (8) and is the critical efficiency beyond which there is no LHV model reproducing all the quantum predictions for all the Bell inequalities, with two settings for q observers ($3 \le q \le n$) and one setting for the other $n-q$ observers, contained in the Mermin inequality. This observation is of practical interest since, e.g., testing each of the $\binom{3}{5}$ 2-2-2-1-1-setting Bell inequalities on a 5-qubit GHZ state requires only 3 spacelike separated regions, while testing the 5-partite Mermin inequality requires 5 spacelike separated regions. When n tends to infinity, η_{crit} tends to $1/2$, reflecting the fact that $I_{0,n,0}$ is a trivial LHV model compatible with the quantum predictions if $\eta \le 1/2$.

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